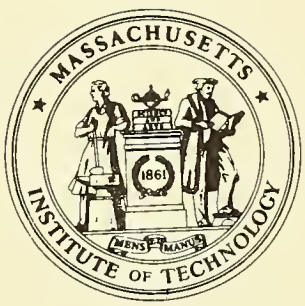

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CALL OPTION PRICING WHEN THE EXERCISE PRICE IS UNCERTAIN, AND THE VALUATION OF INDEX BONDS

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Number 206 July 1977

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The view expressed in this paper are the author's sole responsibility and do not reflect those of the Department of Economics, the Massachusetts Institute of Technology, or the National Science Foundation.

CALL OPTION PRICING WHEN THE EXERCISE PRICE IS UNCERTAIN, AND THE VALUATION OF INDEX BONDS

One of the most attractive features of the recent work on option pricing, ably summarized by Cox and Ross [3] and Smith [8], is the applicability of the option pricing formula to the pricing of other assets. As Black and Scholes [1] point out, corporate stock in a firm which has also issued bonds, can be regarded as a call, with the exercise price being the payment made to the bondholders.

Bonds are typically denominated in nominal terms so that the real value of the payment promised to bondholders is uncertain, even in the event the firm is able to make the payment. The question that then suggests itself is how to value options for which the exercise price is uncertain.

Consider the Black-Scholes (B-S) call pricing formula, where the notation is the same as that of Smith [8]:

$$C = S \cdot N\{\frac{\ln(S/X) + [r + (\sigma^2/2)]T}{\sigma\sqrt{T}}\} - e^{-rT}X \cdot N\{\frac{\ln(S/X) + [r - (\sigma^2/2)]T}{\sigma\sqrt{T}}\} \quad (1)$$

Notation:
- C - call option price
- S - current stock price
- X - exercise price
- r - interest rate
- σ^2 - instantaneous variance of return on the stock
- T - time to expiration
- $N\{\cdot\}$ - cumulative normal distribution function

If all prices are specified in nominal terms -- as they usually are -- then both the variance and the interest rate in the B-S formula should be nominal. That is, the σ^2 variable should be the variance of the nominal rate of return on the stock, and the interest rate variable, r, should be the nominal interest rate on a free-of-default risk nominal bond of maturity T. Were the prices to be specified in real terms, then σ^2 should be the variance of the real rate of return and r should be the real interest rate on a free-of-default risk price-indexed bond of maturity T.

The B-S derivation of (1) is based on an arbitrage argument. If riskless indexed bonds existed, then there would be no difficulty in applying the B-S argument to the valuation of an option with an indexed exercise price. If there were no riskless real bonds, then to use the B-S formula it would be necessary to infer how such bonds would be priced if they existed. More generally, to value options with an uncertain exercise price, it is necessary also to infer how an asset which hedges against changes in the exercise price should be valued. It is natural to use the capital asset pricing model for that purpose.

In this note I discuss the valuation of a call option with an uncertain exercise price, which follows a standard diffusion process. The modified B-S valuation formula which results is compared with the original formula, (1), and the question of the relation between American and European call prices discussed. The modified formula is then used to examine the pricing of indexed and nominal bonds.

1. The Pricing Formula

The standard assumptions of frictionless markets with continuous trading, etc., are set out in Merton [6]. I assume the real stock price[1] has the dynamics

$$\frac{dS}{S} = \alpha_s dt + \sigma_s dz_s \tag{2}$$

where α_s is the instantaneous expected return on the stock and σ_s the instantaneous variance of the return. The stock pays no dividends.

A modified European call on the stock is to be valued. The stated real exercise price of the call at time t is X(t) though the call will be exercisable at time T at X(T), which is not known with certainty at t. For example, X(t) might be equal to $1.17 \times Z(t)$, where Z(t) is the value of a specified stock price index at time t. The actual (uncertain at time $t < T$) exercise price will be $1.17 \times Z(T)$. In this example, the call price is adjusted along with the movements of a stock price index. The payoff from holding the call therefore depends on increases in the price of the stock on which the call is written relative to the specified stock price index, rather than on absolute increases in the price of the particular stock.

The variable X(t) is assumed to follow a diffusion process

$$\frac{dX}{X} = \alpha_x dt + \sigma_x dz_x \tag{3}$$

where α_x is the instantaneous expected rate of increase of the stated exercise price, σ_x^2 is the instantaneous variance of the exercise price, and dz_x is a

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[1] Although the derivation of the pricing formula is carried out with real prices, and thus uses goods as the numeraire, it will become clear that a similar derivation applies for any choice of numeraire.

standard Gauss-Wiener process.[2] The stated exercise price and the stock price have instantaneous correlation coefficient ρ_{sx}:

$$dz_s dz_x = \rho_{sx} dt \tag{4}$$

The problem now is to value a call with currently stated real exercise price X(t). The real call price, C, will be a function of the stock price and the exercise price X(t)[3]. We are interested in the dependence of the call price not only on S and X, but also on the parameters of the stochastic processes (2) and (3).

It will turn out, however, that the call price is also a function of the cost of hedging against changes in the exercise price. By the cost of hedging, I mean the expected real rate of return, r_h, on a security whose stochastic percentage changes in value are perfectly correlated with the stochastic component of the percentage changes in the exercise price. Specifically, the stochastic process for the return on the hedge security is

$$\frac{dH}{H} = r_h dt + \sigma_x dz_x \tag{5}$$

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[2] The distribution of the actual exercise price X(T), conditional on X(t), is lognormal with the expectation of X(T) being

$$E[X(T)/X(t)] = X(t)e^{[\alpha_x - (\sigma_x^2/2)](T-t)}$$

In the body of the paper I discuss only a stochastic process for the proportional change in the exercise price which has independent increments. Footnote 12 discusses alternative specifications of the behavior of the exercise price.

[3] Recall from footnote 1 that the choice of numeraire is not essential, so long as all prices and rates of return are measured in the same numeraire.

Such a security may exist. For instance, in the example given where $X(t) = 1.17\,Z(t)$, with $Z(t)$ a stock price index, it might be possible to construct the portfolio comprising the index. Or, if the exercise price were fixed in nominal terms, so that the real exercise price were perfectly correlated with the purchasing power of money,[4] a nominal bond would constitute the hedge security. In such cases, r_h would be the expected (real) rate of return on the hedge security.

If the hedge security does not exist then the capital asset pricing model can be used to infer what the equilibrium expected rate of return on such a security would be. Let the dynamics of the real market rate of return be given by:

$$\frac{dM}{M} = r_M\,dt + \sigma_M\,dz_M \tag{6}$$

where r_M is the expected (real) market rate of return. Further, define

$$dz_M\,dz_x = \rho_{Mx}\,dt \tag{7}$$

I assume that the parameters r_M, σ_M, and ρ_{Mx}, as well as the risk free real interest rate (or zero beta expected real rate of return) r, to be introduced below, are stationary.

The expected rate of return, r_h, on the hypothetical security that hedges against changes in the exercise price, would be given by:

$$r_h = r + b \tag{8}$$

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[4] The purchasing power of money is the inverse of the general price level.

where b is the risk premium on the hedge security. Then

$$b = \rho_{Mx} \frac{\sigma_x}{\sigma_M} (r_M - r) \tag{9}$$

Note that the cost of hedging against changes in the exercise price is not a function of α_x, the expected rate of change of X; the hedge is insurance only against unanticipated changes in the exercise price.

The derivation of the call pricing formula follows the same lines as the B-S [1] and Merton [6][5] arbitrage derivations. A riskless portfolio comprising the call, the stock, and the hedge security is created. The inclusion of the stock in the portfolio hedges against changes in the stock price, while the inclusion of the hedge security hedges against changes in the exercise price. Since the derivation is very similar to that in [6], it is not necessary to present it.[6] The call pricing formula is:

$$C = S \cdot N\{\frac{\ln(S/X) + [r_h - \alpha_x + (\hat{\sigma}^2/2)]T}{\hat{\sigma}\sqrt{T}}\} - X e^{-(r_h - \alpha_x)T} \cdot N\{\frac{\ln(S/X) + [r_h - \alpha_x - (\hat{\sigma}^2/2)]T}{\hat{\sigma}\sqrt{T}}\} \tag{10}$$

where $\hat{\sigma}^2 = \sigma_s^2 - 2\rho_{sx}\sigma_s\sigma_x + \sigma_x^2$ and t has been taken to be zero. The parameter $\hat{\sigma}^2$ is the instantaneous proportional variance of the change in the ratio (S/X). For a constant exercise price, $\sigma_x = 0$, $\alpha_x = 0$, $r_h = r$ the risk free rate, and (10) reduces to (1), the original B-S formula.

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[5] See especially pp. 164-7 of [6].

[6] A proof that equations (10) and (11) are the call and put pricing formulae respectively when the exercise price is uncertain is available on request from the author.

Similarly, letting $N\{d_1\}$ and $N\{d_2\}$ respectively denote the value of the cumulative normal distribution at the two points specified in (10), the value of a European put with uncertain exercise price is

$$G = Xe^{-(r_h - \alpha_x)T}[1 - N\{d_2\}] - S[1 - N\{d_1\}] \tag{11}$$

We now proceed to check (10) for some properties it ought to have. First, consider using (10), which is expressed in real terms, to value a call option with a fixed nominal exercise price X. Let σ_s^2 be the variance of the real rate of return on the stock. Let Q(t) be the purchasing power of money, so that the real exercise price is XQ, with Q(0) = 1. Assume

$$\frac{dQ}{Q} = \alpha_Q dt + \sigma_Q dz_Q \tag{12}$$

where α_Q is approximately the negative of the expected rate of inflation.[7] Then, in equation (10), $\hat{\sigma}^2$ will be the variance of the rate of change of the nominal stock price S/Q. Further, since r_h is the expected real rate of return on an asset that hedges against changes in the (nominal) exercise price, that asset is a nominal bond. Then $r_h - \alpha_Q$ will be the nominal interest

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[7] "Approximately" because, as shown in Fischer [4], the variance of the proportional change of Q enters the trend term for the diffusion process for $1/Q = P$, the price level. It may be worth pointing out explicitly that if the nominal interest rate on a default-risk-free nominal bond is $\hat{r}$, and the purchasing power of money follows (12), then the expected real rate on that nominal bond is $\hat{r} + \alpha_Q$. In particular, there is no term in σ_Q^2 in the relation between the real and nominal rates. The apparent contrast with [4] arises from the use there of $P = 1/Q$. From Itô's Lemma, as applied in [4], p. 530, it can be shown that $\frac{dP}{P} = (-\alpha_Q + \sigma_Q^2)dt - \sigma_Q dz_Q$. Hence, if I had started here with a stochastic process for $\frac{dP}{P} = \pi dt + s dz$, as in [4], the expected real rate on the nominal bond would be equal to $\hat{r} + s^2 - \pi$. See footnote 9, p. 513 of [4] for the intuitive basis for these relationships.

rate on a free-of-default risk nominal bond. Accordingly in (10), we can replace $\hat{\sigma}^2$ by the variance of the nominal rate of return on the stock, and $r_h - \alpha_x$ by the nominal interest rate. We then have the original B-S formula (1), with all prices and rates of return nominal.

Second, it is clear from (10) that the value of an option to purchase a stock at the stock price is precisely zero. For in that case, $\hat{\sigma} = 0$, and thus the value of $N\{\cdot\}$ at the two places it enters the equation is unity. Further, since the stock pays no dividends, $r_h = \alpha_x$, and since the exercise price is the stock price, $S = X$. Hence $C = 0$ in this case.

Equation (10) may also provide some intuition on the reason the expected rate of appreciation of the stock does not enter the B-S formula. The formula is derived by creating a riskless portfolio. To create the riskless portfolio, it is necessary to hedge against changes in both the stock price and the exercise price. Changes in the stock price are hedged against by holding the stock. The cost of hedging against stock price changes is given by the expected rate of return on the stock. But since the stock is held in the portfolio, the return it provides precisely offsets the cost of hedging against changes in the stock price. Accordingly, the rate of return on the stock does not enter the pricing formula. Changes in the exercise price are hedged against by holding an asset whose return is perfectly correlated with changes in the exercise price. The expected rate of return on the hedge security is r_h. However, the rate at which the exercise price is expected to change, α_x, is not necessarily equal to r_h. Accordingly, the anticipated rate of change of the exercise price does enter the pricing formula. The more rapidly the exercise price is expected to increase (the larger is α_x), the less valuable the call option. It would be surprising if it were otherwise.

If the option were for the right to exchange one non-dividend paying traded asset, with current price, X, for the stock, with current price, S, then $r_h = \alpha_x$, and the expected rate of appreciation of the asset with price X would again not enter the pricing formula.[8]

One of the major attractions of the B-S formula (1) is that its only non-observable argument is σ^2. The modified formula (10) contains variables, namely $\hat{\sigma}^2$, r_h, and α_x, which would in general have to be estimated rather than observed. However, depending on the existence and specifications of a hedge security, some of these variables might be observable. For instance, if all variables in (10) were in real terms, and riskless nominal bonds existed, then the nominal interest rate would replace $r_h - \alpha_x$ and it would be necessary only to estimate $\hat{\sigma}^2$.

The question arises of whether the Merton [6] theorem that an American call on a non-divdend paying stock would not be exercised prematurely holds in the present context.[9] That theorem also implies that an American call on a non-dividend paying stock has the same value as a European call on the stock. If a hedge security (against changes in the exercise price) existed and $r_h - \alpha_x$ were positive (as, for instance, the nominal interest rate is positive), then the Merton dominance argument can be used to show that there would not be premature exercise of an American call.[10] If the hedge security existed and $r_h - \alpha_x$ were negative, the dominance argument is not applicable. An American call might, but would not necessarily, be exercised. The larger α_x, the more likely an American call is to be exercised. Given r_h, so that the unanticipated

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[8] See Margrabe [5]. This paper and Margrabe's were written independently.

[9] We assume in the examples in this paragraph that all variables are measured in real terms.

[10] A proof is available on request from the author.

changes in the exercise price can be hedged away, the situation in which the American call is exercised is one in which the exercise price is expected to increase rapidly. By analogy with the nominal bond as hedge security against changes in the real exercise price of an option with specified nominal exercise price, one can talk imprecisely of the American call being exercised when the "nominal interest rate" on the hedge security is negative.

In the absence of an actual hedge security, the dominance arguments cannot be made.[11] If an American call with uncertain exercise price were created, it would be possible to infer from its value, given $\hat{\sigma}^2$, the market's estimate of $r_h - \alpha_x$. If $r_h - \alpha_x$ were positive, the theory predicts that the option would not be prematurely exercised. If such options were prematurely exercised in sufficient quantities to be certain these were not random disturbances, it would be clear the theory is inadequate.

The discussion of the exercise of an American call with uncertain exercise price is, of course, based on the assumption that r_h and α_x are constant through time. If r_h and α_x could change through time, then an optimal strategy might dictate that an American call not be exercised even at a time when $r_h - \alpha_x$ were temporarily large and negative since the exercise of the call is irreversible and a more favorable opportunity might present itself later -- the exercise price could be expected to fall substantially at some future time, for instance.[12]

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[11] It has been suggested to me that if speculators needed the hedge security, it would be created. However, given the existence of the stock, the modified option with uncertain exercise price may itself be thought of as the hedge security. Thus, the creation of the hedge security could take the form either of a direct hedge, or the modified option.

[12] It is interesting to consider informally the implications of alternative (to (3)) specifications of the dynamics of the exercise price. In valuing a European call, only the terminal distribution of the exercise price, i.e. the distribution of X(T), is of concern. Formulas similar to (10) could accordingly be calculated for stochastic processes other than (3) in which, for instance, the trend term could be a function of the stock price itself, time and other variables, and similarly for the variance. At each point of time the... (continued next page)

II. Indexed and Nominal Bonds

I now use the formulae (1) and (10) to compare the values of indexed and nominal bonds. My interest is in seeing how the possibility of default affects the conclusions reached in [4] on the expected rates of return (and therefore, on discount bonds, the values) of indexed and nominal bonds in market equilibrium. In particular, I want to see whether the intuitive notion that an increase in the correlation between a firm's profits (or the rate of change of the value of the firm) and changes in the price level, increases the value of its indexed relative to its nominal bonds, receives any support from the formal analysis.

Suppose that two firms with identical stochastic processes for their values (of the firm as a whole) issue indexed and nominal pure discount bonds respectively. Assume that the promised payment to the bondholders is X_R for the real bond and the promised nominal payment is X_N for the nominal bond. Let the real payment on the nominal bond be $X_N Q$, where Q, the purchasing power of money, follows the stochastic process (12). Assume further that

$$X_R = X_N e^{\alpha_Q T} \tag{13}$$

This states that the promised real payout is equal to the promised nominal payout, discounted by the expected rate of decline of the purchasing power of money. From footnote 2, it may be seen that the expected payouts are, however, not the same. I want to ask which bond is more valuable and particularly how the correlation between the firms' rate of change of value and the price level

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(footnote 12, continued)... hedge against changes in the exercise price will be valued on the basis of the distribution of X(T). To be sure, the moments of the distribution of X(T) will usually be functions of X(t).

affects the relative values of the bonds. Subscripts R and N will henceforth represent real (or indexed) and nominal, respectively.

Given the current value of the firm, V, and viewing the stock of the firm as an option, we have

$$B = V - C \tag{14}$$

where B is the value of the bond, and C the value of the stock, expressed in terms of the option pricing formulae.

I take the formula (1) to apply to the value of stock in the firm issuing the real bond. The formula (10) is used in valuing the stock in the firm issuing the nominal bonds. Using (1) and (10), and substituting from (13), results in

$$B_R - B_N = S\left[N\left\{\frac{\ln\frac{V}{X_R} + (r_N + \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}\right\} - N\left\{\frac{\ln\frac{V}{X_R} + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right\}\right] \tag{15}$$

$$- X_R\left[e^{-r_N T} N\left\{\frac{\ln\frac{V}{X_R} + (r_N - \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}\right\} - e^{-rT} N\left\{\frac{\ln\frac{V}{X_R} + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right\}\right]$$

In (15), V is the value of the firm, r_N is the expected real rate of return on default-risk-free nominal bonds (equal to the nominal interest rate plus α_Q), r is the real interest rate on default-risk-free index bonds, $\hat{\sigma}^2$ is the variance of the rate of change of the nominal value of the firm, and σ^2 is the variance of the rate of change of the firm's real value.

From (15), it can be seen that the value of the nominal bond may differ from that of the real bond as a result of differences between r and r_N, and between σ^2 and $\hat{\sigma}^2$. Differences between r and r_N reflect free-of-default risk differences in expected real returns on the two bonds, while differences between σ^2 and $\hat{\sigma}^2$ reflect default risk differences.

Since the value of an option is an increasing function of both the interest rate and the variance of the rate of return on the stock, the indexed bond will tend to command a higher price if $r < r_N$ and if $\sigma^2 < \hat{\sigma}^2$. For r to be less than r_N, the purchasing power of money has to be positively correlated with the market rate of return -- or the market rate of return has to be inversely correlated with the rate of inflation.[13] Recent empirical evidence suggests that stock market returns are indeed negatively correlated with unanticipated changes in the price level.[14] For σ^2 to be less than $\hat{\sigma}^2$, the variance of changes in the real value of the firm has to be smaller than the variance of changes in the nominal value of the firm. A sufficient condition for this is that ρ_{Qv} be negative, or that changes in the real value of this particular firm be positively correlated with unanticipated inflation.

Accordingly, an indexed bond in a given firm will tend to be more valuable than a nominal bond in the same firm -- given the payments to the bondholders conditional on no default specified in (13) -- first, the lower the real interest rate on free-of-default risk indexed bonds compared with the expected rate of return on free-of-default risk nominal bonds, and second, the greater the correlation between changes in the real value of the firm and unanticipated inflation. Each of these conclusions is intuitively plausible. The second conclusion provides support for the notion discussed in the opening paragraph of this section.

Of course, given the Modigliani-Miller theorem, the value of the stock in the two firms will be affected in precisely the opposite direction by differences between r and r_N, and σ^2 and $\hat{\sigma}^2$ respectively.

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[13] This conforms with the conclusions in [4].

[14] See, for example, Bodie [2] and Nelson [7].

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